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Long Time Anderson Localization for the Nonlinear Random Schrödinger Equation

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Abstract We prove long time Anderson localization for the nonlinear random Schrödinger equation for arbitrary ℓ^2 initial data, hence giving an answer to a widely debated question in the physics community. The proof uses a Birkhoff normal form type transform to create a barrier where there is essentially no propagation. One of the new features is that this transform is in a small neighborhood enabling us to treat "rough" data, where there are no moment conditions. The formulation of the present result is inspired by the RAGE theorem.

Keywords Anderson localization · Birkhoff normal form

1 Introduction

We consider the lattice nonlinear random Schrödinger equation in $1-d$:

$$i\dot{q}_j = v_j q_j + \epsilon_1(q_{j-1} + q_{j+1}) + \epsilon_2 |q_j|^2 q_j, \quad j \in \mathbb{Z}, \tag{1.1}$$

where $V = \{v_j\}$ is a family of independent identically distributed (i.i.d.) random variables in [0, 1] with uniform distribution and $\epsilon_1, \epsilon_2 > 0$. The cubic nonlinearity models the particle-particle interaction. In physics, (1.1) is important in the study of Bose-Einstein condensation and nonlinear optics.

When $\epsilon_2 = 0$, (1.1) is the well studied Anderson tight binding model, where it is known [10] that for all ϵ_1, the Schrödinger operator

$$H = \epsilon_1 \Delta + V \quad \text{on } \ell^2(\mathbb{Z}), \tag{1.2}$$

To J. Fröhlich and T. Spencer on their sixtieth birthdays.

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where

$$\begin{aligned} \Delta_{jk} &= 1, \quad |i-j|_{\ell^1} = 1, \\ &= 0, \quad \text{otherwise}, \end{aligned} \tag{1.3}$$

almost surely has pure point spectrum with exponentially localized eigenfunctions. In $d \geq 2$, it is known [1, 9, 13] that for $0 \leq \epsilon_1 \ll 1$ almost surely the spectrum is pure point with exponentially localized eigenfunctions. This is called Anderson localization (A.L.). By the RAGE theorem [3, 8, 12] (cf. also [7]) pure point spectrum is equivalent to the following statement:
for all initial datum $\{q_j(0)\}$ in ℓ^2 and $\delta > 0$, there exists j_0 such that

$$\sup_{t\in\mathbb{R}} \sum_{|j|>j_0} |q_j(t)|^2 \leq \delta. \tag{1.4}$$

When $\epsilon_2 \neq 0$, spectral theory is no longer available. However we can still retain (1.4) as a criterion for the nonlinear equation (1.1). In other words, we are interested in the persistence of A.L. with nonlinearity. In this paper, we prove that for *any* ℓ^2 initial datum, A.L. persists for polynomially long (in $(\epsilon_1+\epsilon_2)^{-1}$) time, hence giving an answer to a widely debated question in the physics community.

More precisely, we work in ℓ^2, the space for the linear theory. This is possible as it is easily seen that (1.1) has a global solution in ℓ^2 and the ℓ^2 norm of the solution $\{q_j(t)\}$ is conserved, i.e.,

$$\sum_{j\in\mathbb{Z}} |q_j(t)|^2 = \sum_{j\in\mathbb{Z}} |q_j(0)|^2, \quad \text{for all } t \in \mathbb{R}. \tag{1.5}$$

Let $\epsilon = \epsilon_1 + \epsilon_2$. We prove

Theorem 1.1 *Given $\delta > 0$, $A > 1$ for all initial datum $\{q_j(0)\}_{j\in\mathbb{Z}} \in \ell^2$, let $j_0 \in \mathbb{N}$ be such that*

$$\sum_{|j|>j_0} |q_j(0)|^2 < \delta. \tag{1.6}$$

Then there exist $C = C(A) > 0$, $\epsilon(A) > 0$ and $N = N(A) > A^2$ such that for all $t \leq (\delta/C)\epsilon^{-A}$,

$$\sum_{|j|>j_0+N} |q_j(t)|^2 < 2\delta, \tag{1.7}$$

with probability

$$1 - \exp\left(-\frac{j_0}{N} e^{-2N\epsilon^{\frac{1}{CA}}}\right), \tag{1.8}$$

provided $0 < \epsilon < \epsilon(A)$.

We interpret the above result as long time Anderson localization for the nonlinear random Schrödinger equation (1.1). The main difference with (1.4) is that in the nonlinear case, the displacement of the wave front N is *not* uniformly bounded in t. We believe that this is the true behavior of the nonlinear equation, which we will formulate later as a conjecture.

The proof of Theorem 1.1 uses Birkhoff normal form type transformations. The main feature of this normal form is that it contains barriers centered at some $\pm j_0 \in \mathbb{Z}$, $j_0 > 1$ of width N, where the terms responsible for propagation are small $\sim \epsilon^A$. This is similar to the normal form transform in [5]. The fact that the transformation is only in a small neighborhood enables us to treat ℓ^2 data. They are "rough" data when viewing $j \in \mathbb{Z}$ as a Fourier index.

This normal form is different from the usual Birkhoff normal form used in nonlinear PDE's, cf. e.g., [4], where the perturbation needs to be of order at least 3 and moreover one typically needs smooth initial data, which in the present context means that $\{q_j(0)\}$ such that $\sum j^{2s}|q_j(0)|^2 = 1$ for $s > s_0 \gg 1$. The ϵ_1 term in the Hamiltonian (cf. (2.2)) is of order 2 here, which introduces some subtleties in the measure estimates. The present method seems particularly suited to treat nonlinear lattice Schrödinger equations, where the interactions are typically short range.

We now comment on a fine point, namely the small parameter ϵ_1 in (1.1), which was not needed in $1-d$ to prove A.L. for the linear equation. The reason we need it for the nonlinear equation is because we need to exclude certain potential configurations in addition to what is needed for A.L. This is in order to avoid small denominators which correspond to new resonances generated by the nonlinearity. Since this exclusion is *a postiori*, had we used the bases provided by the eigenfunctions we would have needed precise information on how the eigenfunctions vary as the potential varies. To our knowledge, this does not seem to be available in the existing literature.

The above theorem raises the natural question of the limit as $t \to \infty$ (independent of ϵ_1 and ϵ_2). In [6], time quasi-periodic solutions were constructed in all dimensions for small ϵ_1 and ϵ_2. (Previously a special type of time periodic solutions where there is only the basic frequency was constructed in [2].) Certainly in this case (1.4) remains valid as $t \to \infty$. The validity or invalidity of (1.4) as $t \to \infty$ for more general initial data remains essentially an open problem.

However from the observation that in Theorem 1.1, roughly speaking $\frac{N}{(\log t)^p} \sim$ constant for appropriate $p \geq 2$, it is natural to make the following

Conjecture As $t \to \infty$, the displacement of the wave front is of order t^{0^+} (possibly logarithmic).

Presently we do not know of a natural nonlinear system that exhibits such Arnold diffusion type of behavior. However in the case of the linear Schrödinger equation on the circle with an arbitrary smooth time dependent potential, it was proven in [14] that the growth of Sobolev norms, i.e., the diffusion into higher Fourier frequencies is at most logarithmic in t. On the other hand, it was proven in [11] that with the additional assumption that the time dependence is random, the Sobolev norms are unbounded with probability 1. Therefore we have both an upper and lower bound. The results in [11, 14] lend further credence to the conjecture.

2 Structure of the transformed Hamiltonian

We recast (1.1) as a Hamiltonian equation:

$$i\dot{q}_j = 2\frac{\partial H}{\partial \overline{q}_j}, \tag{2.1}$$

with the Hamiltonian

$$H(q,\bar{q})=\frac{1}{2}\left(\sum_{j\in\mathbb{Z}}v_j|q_j|^2+\epsilon_1\sum_{j\in\mathbb{Z}}(\bar{q}_jq_{j+1}+q_j\bar{q}_{j+1})+\frac{1}{2}\epsilon_2\sum_{j\in\mathbb{Z}}|q_j|^4\right). \tag{2.2}$$

As mentioned earlier, the ℓ^2 norm of the solution $\{q_j(t)\}$ is conserved, i.e.

$$\sum_{j\in\mathbb{Z}}|q_j(t)|^2=\sum_{j\in\mathbb{Z}}|q_j(0)|^2,\quad \forall t\in\mathbb{R}. \tag{2.3}$$

In order to prove (1.7), we need to control the time derivative of the truncated sum of higher modes

$$\frac{d}{dt}\sum_{|j|>j_0}|q_j(t)|^2. \tag{2.4}$$

As in [5], we will use the random potential $V=\{v_j\}_{j\in\mathbb{Z}}$ to obstruct energy transfer from low to high modes by creating "zones" in $\mathbb{Z}$, where the only mode coupling term is of order $O(\epsilon^A)$, where $\epsilon=\epsilon_1+\epsilon_2$ as before. This construction is achieved by invoking the usual process of symplectic transformations.

In what follows, we will deal extensively with monomials in q_j. So we first introduce some notations. Rewrite any monomials in the form:

$$\prod_{j\in\mathbb{Z}}q_j^{n_j}\bar{q}_j^{n'_j}.$$

Let $n=\{n_j,n'_j\}_{j\in\mathbb{Z}}\in\mathbb{N}^{\mathbb{Z}}\times\mathbb{N}^{\mathbb{Z}}$. We will use three notations: support, diameter and degree:

$$\begin{aligned}\operatorname{supp} n&=\{j\,|\,n_j\neq 0 \text{ or } n'_j\neq 0\},\\ \Delta(n)&=\operatorname{diam}\{\operatorname{supp} n\},\\ |n|&=\sum_{j\in\operatorname{supp} n}(n_j+n'_j).\end{aligned}$$

If $n_j=n'_j$ for all $j\in\operatorname{supp} n$, then the monomial is called resonant. Otherwise it is called non-resonant. Note that non-resonant monomials contribute to the truncated sum in (2.4), while resonant ones do not.

To control the sum in (2.4), we will transform H in (2.2) to H' of the form:

$$H'=\frac{1}{2}\sum_{j\in\mathbb{Z}}(v_j+w_j)|q_j|^2 \tag{2.5}$$

$$+\sum_{n\in\mathbb{N}^{\mathbb{Z}}\times\mathbb{N}^{\mathbb{Z}}}c(n)\prod_{\operatorname{supp} n}q_j^{n_j}\bar{q}_j^{n'_j} \tag{2.6}$$

$$+\sum_{n\in\mathbb{N}^{\mathbb{Z}}\times\mathbb{N}^{\mathbb{Z}}}d(n)\prod_{\operatorname{supp} n}|q_j|^{2n_j} \tag{2.7}$$

$$+\,O(\epsilon^A), \tag{2.8}$$

where (2.6) consists of non-resonant monomials ($n_j \neq n'_j$ for some j), (2.7) consists of resonant monomials of degree at least 4. Note that

$$\sum_{\operatorname{supp} n} n_j = \sum_{\operatorname{supp} n} n'_j \tag{2.9}$$

for non-resonant monomials in (2.6), which is a general feature of polynomial Hamiltonian. The coefficients $c(n)$ and $d(n)$ satisfy the bound

$$\forall n, \quad |c(n)| + |d(n)| < \exp\left(-\rho\{\Delta(n) + |n| - 2\} \log \frac{1}{\epsilon}\right), \quad \rho > \frac{1}{10}. \tag{2.10}$$

The transformed Hamiltonian H' will manifest an “energy barrier”. More precisely, we require that

$$|c(n)| < \epsilon^A, \quad \text{if } \operatorname{supp} n \cap \{[-b, -a] \cup [a, b]\} \neq \emptyset, \tag{2.11}$$

where a and b satisfy

$$\left[j_0 - \frac{N}{2}, j_0 + \frac{N}{2}\right] \subset [a, b] \subset [j_0 - N, j_0 + N], \tag{2.12}$$

and N is an integer depending on A.

In (2.5), w_j and all coefficients $c(n), d(n)$ depend on V. Let $W = \{w_j\}_{j \in \mathbb{Z}}$. We require that

$$|\nabla_V c(n)| + |\nabla_V d(n)| < 1 \tag{2.13}$$

and

$$\left\| \frac{\partial W}{\partial V} \right\|_{\ell^2 \to \ell^2} < \epsilon^{\frac{1}{40}}. \tag{2.14}$$

The transformation from H to H' will be achieved by a finite step iterative process. Let H_s, Γ_s be the Hamiltonian and the transformation at step s, $H_{s+1} = H_s \circ \Gamma_s$. At each step s, Γ_s is the symplectic transformation generated by an appropriate polynomial Hamiltonian F. H_{s+1} is the time-1 map, computed by using a convergent Taylor series of successive Poisson brackets of H_s and F, i.e.

$$H_{s+1} = H_s \circ \Gamma_s = \{H_s, F\} + \frac{1}{2!}\{\{H_s, F\}, F\} + \cdots,$$

where the Poisson bracket $\{H_s, F\}$ is defined by

$$\{H_s, F\} = \sum_{j \in \mathbb{Z}} \frac{\partial H_s}{\partial \bar{q}_j} \frac{\partial F}{\partial q_j} - \frac{\partial H_s}{\partial q_j} \frac{\partial F}{\partial \bar{q}_j}.$$

It is important to remark that our construction only involves the modes $j \in \mathbb{Z}$ for which $||j| - j_0| \leq N$. So, if

$$\operatorname{supp} n \cap \{[-j_0 - N, -j_0 + N] \cup [j_0 - N, j_0 + N]\} \neq \emptyset$$

in the sum of (2.6–2.7), then we have by the fact that $\Delta(n) \leq 1$ for all the terms in H that

$$\operatorname{supp} n \subset [-j_0 - N - 1, -j_0 + N + 1] \cup [j_0 - N - 1, j_0 + N + 1],$$

which together with (2.10) implies that we can assume that

$$\Delta(n) < 20A \quad \text{and} \quad |n| < 20A \tag{2.15}$$

for the terms with $\operatorname{supp} n \cap \{[-j_0 - N, -j_0 + N] \cup [j_0 - N, j_0 + N]\} \neq \emptyset$, the other terms can be captured by (2.8) in H'. On the other hand, one has

$$|c(n)|,\ |d(n)| \leq \epsilon, \qquad \Delta(n) \leq 1 \tag{2.16}$$

for the terms with $\operatorname{supp} n \cap \{[-j_0 - N, -j_0 + N] \cup [j_0 - N, j_0 + N]\} = \emptyset$.

In addition, $w_j = 0$ unless $||j| - j_0| \leq N + 1$, which together with (2.14) implies that the frequency modulation map $V \to \widetilde{V} = V + W$ satisfies

$$e^{-2N\epsilon^{\frac{1}{40}}} < (1 - \epsilon^{\frac{1}{40}})^{2N+2} < \left|\det \frac{\partial \widetilde{V}}{\partial V}\right| < (1 + \epsilon^{\frac{1}{40}})^{2N+2} < e^{2N\epsilon^{\frac{1}{40}}}. \tag{2.17}$$

The non-resonance estimates in Sect. 3 on symplectic transforms are expressed in terms of $\widetilde{V}$. These non-resonance estimates will be translated into probabilistic estimates in V by using (2.17).

3 Analysis and Estimates of the Symplectic Transformations

We now construct the symplectic transformation Γ so that the transformed Hamiltonian $H' = H \circ \Gamma$ satisfies (2.10)–(2.14). It is achieved by a finite step induction. At the first step: $s = 1$

$$H_1 = H = \frac{1}{2}\left(\sum_{j\in\mathbb{Z}} v_j |q_j|^2 + \epsilon_1 \sum_{j\in\mathbb{Z}} (\bar{q}_j q_{j+1} + q_j \bar{q}_{j+1}) + \frac{1}{2}\epsilon_2 \sum_{j\in\mathbb{Z}} |q_j|^4\right).$$

Let η_j denote the canonical basis of $\mathbb{Z}$, (2.10)–(2.12) are satisfied with

$$\begin{aligned} c(n) = c(\eta_j \times \eta_{j+1}) &\leq \frac{\epsilon}{2}, \quad |n| = 2,\ \Delta(n) = 1,\ \operatorname{supp} n = \{j, j+1\}, \\ &= 0 \quad \text{otherwise}, \\ d(n) = d(\eta_j \times \eta_{j+1}) &\leq \frac{\epsilon}{4}, \quad |n| = 4,\ \Delta(n) = 0,\ \operatorname{supp} n = \{j\}, \\ &= 0 \quad \text{otherwise}. \end{aligned}$$

Equation (2.13) is trivially satisfied with

$$|\nabla_V(c)| = |\nabla_V(d)| = 0,$$

and so is (2.14):

$$W = 0, \qquad \frac{\partial W}{\partial V} = 0.$$

Assume that we have obtained at step s, the Hamiltonian H_s in the form (2.5–2.8) satisfying the counterpart of (2.10)–(2.14) at step s, which we denote as $(2.10)_s$–$(2.14)_s$ and which we will make explicit below. Our aim is to produce H_{s+1} possessing the corresponding properties at step $s+1$. In what follows, ϵ always denotes a sufficiently small constant depending only on A. $(2.10)_s$–$(2.12)_s$ state that

$$|c(n)| + |d(n)| < \exp\left(-\rho_s\{\Delta(n) + |n| - 2\}\log\frac{1}{\epsilon}\right), \tag{3.1}$$

with $\rho_s > \frac{1}{10}$, moreover

$$|c(n)| < \delta_s \quad \text{if } \operatorname{supp} n \cap \{[-b_s, -a_s] \cup [a_s, b_s]\} \neq \emptyset, \tag{3.2}$$

with

$$\left[j_0 - \frac{N}{2}, j_0 + \frac{N}{2}\right] \subset [a_s, b_s] \subset [j_0 - N, j_0 + N], \tag{3.3}$$

and δ_s is defined inductively as

$$\delta_1 = \frac{\epsilon}{2}, \qquad \delta_s = \delta_{s-1}^{\frac{19}{10}} + \epsilon^{\frac{1}{20}}\delta_{s-1}, \quad s \geq 2. \tag{3.4}$$

We remark that the first term in (3.4) comes from Poisson brackets of polynomials with coefficients c, while the second one comes from Poisson brackets of polynomials with coefficients c and d.

We satisfy (3.2) at step $s+1$ constructively by removing those $c(n)$ with $\delta_{s+1} < |c(n)| < \delta_s$, $\operatorname{supp} n \cap \{[-b_s, -a_s] \cup [a_s, b_s]\} \neq \emptyset$ and a corresponding reduction of

$$[-b_s, -a_s] \cup [a_s, b_s] \quad \text{to} \quad [-b_{s+1}, -a_{s+1}] \cup [a_{s+1}, b_{s+1}]$$

with $a_{s+1} > a_s, b_{s+1} < b_s$, so that

$$|c(n)| \leq \delta_{s+1} \quad \text{if } \operatorname{supp} n \cap \{[-b_{s+1}, -a_{s+1}] \cup [a_{s+1}, b_{s+1}]\} \neq \emptyset. \tag{3.5}$$

We proceed as follows. Denoting in H_s (2.5)–(2.8),

$$\widetilde{v}_j = v_j + w_j^{(s)} \quad \text{and} \quad H_0 = \sum_{j\in\mathbb{Z}} \widetilde{v}_j |q_j|^2, \tag{3.6}$$

we define, following the standard approach

$$H_{s+1} = H_s \circ \Gamma_s, \tag{3.7}$$

where Γ_s is the symplectic transformation obtained from the Hamiltonian function

$$F = \sum_{\operatorname{supp} n \subset [-b_s, -a_s] \cup [a_s, b_s], |c(n)| > \delta_{s+1}} \frac{c(n)}{\sum (n_j - n'_j)\widetilde{v}_j} \prod q_j^{n_j} \bar{q}_j^{n'_j}.$$

Here we need to impose the small divisor condition

$$\left|\sum (n_j - n'_j)\widetilde{v}_j\right| > \delta_s^{\frac{1}{100s^2}}, \tag{3.8}$$

which will lead to measure estimates of this construction in Sect. 4.

Recall that H_{s+1} is the time-1 map and by Taylor series:

$$
\begin{aligned}
H_{s+1} = H_s \circ \Gamma_s = H_0 + (2.6) + \{H_0, F\} + (2.7) & \\
+ \{(2.6), F\} + \frac{1}{2!}\{\{(2.6), F\}, F\} + \cdots & \qquad (3.9)\\
+ \{(2.7), F\} + \frac{1}{2!}\{\{(2.7), F\}, F\} + \cdots & \qquad (3.10)\\
+ (2.8) \circ \Gamma_F. &
\end{aligned}
$$

Note that

$$
\{H_0, F\} = \sum_{\operatorname{supp} n \subset [-b_s, -a_s] \cup [a_s, b_s], |c(n)| > \delta_{s+1}} c(n) \prod q_j^{n_j} \bar{q}_j^{n'_j},
$$

and if $|c(n)| > \delta_{s+1}$, then by (3.1)

$$
\Delta(n) < 10 \frac{\log \frac{1}{\delta_{s+1}}}{\log \frac{1}{\epsilon}}.
$$

Define

$$
a_{s+1} = a_s + 20 \frac{\log \frac{1}{\delta_{s+1}}}{\log \frac{1}{\epsilon}}, \qquad b_{s+1} = b_s - 20 \frac{\log \frac{1}{\delta_{s+1}}}{\log \frac{1}{\epsilon}}. \tag{3.11}
$$

Then $\{H_0, F\}$ removes in (2.6) all monomials for which $|c(n)| > \delta_{s+1}$ and $\operatorname{supp} n \cap \{[-b_{s+1}, -a_{s+1}] \cup [a_{s+1}, b_{s+1}]\} \neq \emptyset$. Thus, (2.6) + $\{H_0, F\}$ satisfies (3.5). Note that a_{s+1} and b_{s+1} do not shrink to j_0 for N large enough depending only on A (cf. (3.25)). We next prove that (3.9, 3.10) satisfy $(3.1, 3.2)_{s+1}$.

Monomials in (3.9)

We begin with the first two Poisson brackets. We rewrite them as

$$
\{(2.6), F\} = \sum_\mu g_1(\mu) \prod q_j^{\mu_j} \bar{q}_j^{\mu'_j},
$$

$$
\frac{1}{2!}\{\{(2.6), F\}, F\} = \sum_\mu g_2(\mu) \prod q_j^{\mu_j} \bar{q}_j^{\mu'_j}.
$$

The Poisson bracket $\{(2.6), F\}$ produces monomials of the form

$$
\begin{aligned}
&\left\{ \prod q_j^{m_j} \bar{q}_j^{m'_j}, \prod q_j^{n_j} \bar{q}_j^{n'_j} \right\} \\
&\quad = \sum (m_k n'_k - m'_k n_k) q_k^{m_k + n_k - 1} \bar{q}_k^{m'_k + n'_k - 1} \prod_{j \neq k} q_j^{m_j + n_j} \bar{q}_j^{m'_j + n'_j},
\end{aligned} \tag{3.12}
$$

with the coefficient

$$
\frac{c(m)c(n)}{\sum (n_j - n'_j) \widetilde{v}_j},
$$

where

$$\operatorname{supp} n \subset [a_s, b_s] \cup [-b_s, -a_s], \qquad \operatorname{supp} m \cap \operatorname{supp} n \neq \emptyset.$$

Hence $\operatorname{supp} m \cap \{[a_s, b_s] \cup [-b_s, -a_s]\} \neq \emptyset$. Then it follows from (3.2) that

$$|c(m)|, |c(n)| < \delta_s. \tag{3.13}$$

The monomials in (3.12) corresponding to multi-index μ satisfy

$$\Delta(\mu) \leq \Delta(m) + \Delta(n), \quad |\mu| = |m| + |n| - 2. \tag{3.14}$$

The number of realizations of a fixed monomials $\prod q_j^{\mu_j} \bar{q}_j^{\mu'_j}$ is bounded by

$$2^{|\mu|}(\Delta(m) \wedge \Delta(n)) < 2^{|\mu|}(\Delta(m) + \Delta(n)). \tag{3.15}$$

Summing up (3.13–3.15), we get by using the small divisor bound (3.27) that

$$|g_1(\mu)| \leq 2^{|\mu|}(|\mu| + 2)^2 \delta_s^{-\frac{1}{100s^2}} |c(m)||c(n)|(\Delta(m) + \Delta(n)). \tag{3.16}$$

Define

$$\rho_{s+1} = \rho_s \left(1 - \frac{1}{10s^2}\right).$$

Then we get by using (3.1), (3.13) and (3.14) that

$$\begin{aligned} |g_1(\mu)| \leq 2^{|\mu|}(|\mu| + 2)^2 |\log \delta_s^2| \delta_s^{-\frac{1}{100s^2}} \delta_s^{\frac{2}{10s^2}} \\ \times \exp\left(-\rho_{s+1}\{\Delta(\mu) + |\mu| - 2\} \log \frac{1}{\epsilon}\right), \end{aligned} \tag{3.17}$$

where we used (3.1) to bound $\Delta(m), \Delta(n)$ in terms of $c(m), c(n)$. Note that by (2.15) and (3.14)

$$|\mu| < 40A,$$

and by (3.4)

$$\log \delta_{s+1}^{-1} \approx s \log \frac{1}{\epsilon} \tag{3.18}$$

which implies that we can terminate the construction at step $s_* \sim A$ such that $\delta_{s_*} < \epsilon^A$. Thus (3.17) gives that

$$g_1(\mu) < \delta_s^{\frac{1}{20s^2}} \exp\left(-\rho_{s+1}\{\Delta(\mu) + |\mu| - 2\} \log \frac{1}{\epsilon}\right), \tag{3.19}$$

for ϵ sufficiently small depending only on A.

We now turn to the estimate of $g_2(\mu)$. A fixed monomial $\prod q_j^{\mu_j} \bar{q}_j^{\mu'_j}$ in $\{\{(2.6), F\}, F\}$ is now the confluence of 3 sources, denoted by m, n, p with

$$|\mu| = |m| + |n| + |p| - 4.$$

Continuing the previous terminology, the coefficient is

$$c(m)c(n)c(p) \quad \text{with} \;\; |c(m)|, |c(n)|, |c(p)| < \delta_s,$$

and the prefactor is a sum of terms of the form

$$(m_k n_k' - m_k' n_k)[(m_j + n_j)p_\ell' - (m_j' + n_j')p_\ell] \quad \text{if } j \neq k \quad \text{or}$$
$$(m_k n_k' - m_k' n_k)[(m_k + n_k - 1)p_\ell' - (m_k' + n_k' - 1)p_\ell].$$

Hence the prefactor is bounded by $(|\mu| + 4)^4$. The entropy is bounded by

$$\sum_{|w|=2}^{|\mu|} 2^{|w|}(\Delta(m) \wedge \Delta(n))2^{|\mu|}(\Delta(w) \wedge \Delta(p)) \leq [2^{|\mu|}(\Delta(m) + \Delta(n) + \Delta(p))]^2.$$

Therefore we have

$$\begin{aligned} g_2(\mu) &\leq \frac{1}{2!}(\delta_s^{-\frac{1}{100s^2}})^2(|\mu| + 4)^4[2^{|\mu|}(\Delta(m) + \Delta(n) + \Delta(p))]^2 \delta_s^{\frac{3}{10s^2}} \\ &\quad \times \exp\left(-\rho_{s+1}\{\Delta(\mu) + |\mu| - 2\} \log \frac{1}{\epsilon}\right) \\ &\leq \frac{1}{2!}(\delta_s^{-\frac{1}{100s^2}})^2[2^{|\mu|}(|\mu| + 4)^2]^2 |\log \delta_s^3|^2 \delta_s^{\frac{3}{10s^2}} \exp\left(-\rho_{s+1}\{\Delta(\mu) + |\mu| - 2\} \log \frac{1}{\epsilon}\right) \\ &< (\delta_s^{\frac{1}{20s^2}})^2 \exp\left(-\rho_{s+1}\{\Delta(\mu) + |\mu| - 2\} \log \frac{1}{\epsilon}\right). \end{aligned} \tag{3.20}$$

From (3.19, 3.20), the structure of the estimates on the Poisson brackets in (3.9) is clear and we obtain that the $\prod q_j^{\mu_j} \bar{q}_j^{\mu_j'}$ factor in (3.9) is bounded by

$$g(\mu) < \exp\left(-\rho_{s+1}\{\Delta(\mu) + |\mu| - 2\} \log \frac{1}{\epsilon}\right). \tag{3.21}$$

Furthermore, we also have

$$g(\mu) < \delta_s^{2 - \frac{1}{50s^2}} < \delta_s^{\frac{19}{10}}. \tag{3.22}$$

Monomials in (3.10)

We rewrite (3.10) as

$$(3.10) = \sum_\mu \gamma(\mu) \prod q_j^{\mu_j} \bar{q}_j^{\mu_j'}.$$

Similar to the proof of (3.21) and (3.22), we get by using the fact that $|d(n)| \leq \epsilon^{\frac{1}{10}}$ that

$$\gamma(\mu) < \delta_s^{\frac{1}{20s^2}} \exp\left(-\rho_{s+1}\{\Delta(\mu) + |\mu| - 2\} \log \frac{1}{\epsilon}\right), \tag{3.23}$$

$$\gamma(\mu) < \epsilon^{\frac{1}{20}} \delta_s. \tag{3.24}$$

Summing up (3.21)–(3.24), we conclude that H_{s+1} satisfies $(3.1, 3.2)_{s+1}$. We now check $(3.3)_{s+1}$ for the interval $[a_s, b_s]$. We get by (3.11) that

$$|a_s - a_{s+1}| + |b_s - b_{s+1}| \leq 20 \frac{\log \frac{1}{\delta_{s+1}}}{\log \frac{1}{\epsilon}}.$$

Let $s_* \sim A$ be such that $\delta_{s_*} < \epsilon^A$. Then we have

$$|a_1 - a_{s_*}| + |b_1 - b_{s_*}| \leq \frac{20}{\log \frac{1}{\epsilon}} \sum_{t \leq s_*} \log \frac{1}{\delta_t} \lesssim A^2, \tag{3.25}$$

and $(3.3)_{s+1}$ will hold from $a_1 = j_0 - N, b_1 = j_0 + N$, if $N \gg A^2$.

Finally, let us check $(2.13, 2.14)_{s+1}$ for H_{s+1}. In H_0, we need to add resonant quadratic terms produced in (3.9, 3.10). Denoting these terms by $w_j^{(s)}$, $\widetilde{v}_j$ is then perturbed to

$$\widetilde{\widetilde{v}}_j = \widetilde{v}_j + w_j^{(s)},$$

where $w_j^{(s)}$ by construction satisfies

$$|w_j^{(s)}| < \delta_{s+1}. \tag{3.26}$$

Therefore, all non-resonance conditions imposed so far can be replaced by

$$\left| \sum (n_j - n_j') \widetilde{\widetilde{v}}_j \right| > \delta_s^{\frac{1}{100 s^2}}, \tag{3.27}$$

for all $t \leq s$ and n satisfying

$$\operatorname{supp} n \subset [a_t, b_t] \cup [-b_t, -a_t], \qquad \Delta(n) < 10 \frac{\log \frac{1}{\delta_{s+1}}}{\log \frac{1}{\epsilon}}.$$

Finally, we check the V dependence for $g(\mu), \gamma(\mu)$. We have

$$|\nabla_V g_1(\mu)| + |\nabla_V \gamma_1(\mu)| < 2^{|\mu|} (|\mu| + 2)^2 (\Delta(m) + \Delta(n)) \left| \nabla_V \frac{c(m) c(n)}{\sum (n_j - n_j') \widetilde{v}_j} \right|. \tag{3.28}$$

Using (2.13) and (2.14), we have

$$\begin{aligned}
&\left| \nabla_V \frac{c(m) c(n)}{\sum (n_j - n_j') \widetilde{v}_j} \right| \\
&\quad \leq \big[(|\nabla_V c(m)| + |\nabla_V d(m)|) |c(n)| + (|c(m)| + |d(m)|) |\nabla_V c(n)| \big] \left| \sum (n_j - n_j') \widetilde{v}_j \right|^{-1} \\
&\qquad + 2 N^{\frac{1}{2}} |n| (|c(m)| + |d(m)|) |c(n)| \left| \sum (n_j - n_j') \widetilde{v}_j \right|^{-2} \| D \widetilde{V} \|_{\ell^2 \to \ell^2} \\
&\quad < 2 \delta_s^{-\frac{1}{100 s^2}} (\delta_s + \epsilon^{\frac{1}{10}}),
\end{aligned}$$

which together with (3.27) gives that

$$|\nabla_V g_1(\mu)| + |\nabla_V \gamma_1(\mu)| < \delta_s^{\frac{2}{3}} + \epsilon^{\frac{1}{15}}.$$

The higher order brackets can be treated similarly and we obtain

$$|\nabla_V g(\mu)| + |\nabla_V \gamma(\mu)| < \delta_s^{\frac{1}{2}} + \epsilon^{\frac{1}{20}}.$$

In particular, $(2.13)_{s+1}$ holds and morover from Schur's lemma

$$\left\| \frac{\partial W^{(s)}}{\partial V} \right\|_{\ell^2 \to \ell^2} \lesssim (\delta_s^{\frac{1}{2}} + \epsilon^{\frac{1}{20}}) \frac{\log \frac{1}{\delta_{s+1}}}{\log \frac{1}{\epsilon}} < \delta_s^{\frac{1}{3}} + \epsilon^{\frac{1}{30}}$$

as $\Delta(n) < 10 \frac{\log \frac{1}{\delta_{s+1}}}{\log \frac{1}{\epsilon}}$ and $\log \delta_{s+1}^{-1} \approx s \log \frac{1}{\epsilon}$, $s \leq s_* \sim A$. Since $W = \sum_{s=1}^{s_*} W^{(s)}$, $(2.14)_{s+1}$ remains valid along the process.

4 Estimates on the Measure

Recall that the estimates on the symplectic transformations in Sect. 3 depend on the non-resonance condition

$$\left| \sum (n_j - n'_j) \widetilde{v}_j \right| > \delta_s^{\frac{1}{100s^2}}, \tag{4.1}$$

where n satisfies

$$\operatorname{supp} n \subset [a_s, b_s] \cup [-b_s, -a_s], \qquad \Delta(n) < 10 \frac{\log \frac{1}{\delta_{s+1}}}{\log \frac{1}{\epsilon}}, \qquad |n| \leq 20A, \tag{4.2}$$

Moreover, $\widetilde{v}_j$ can be replaced by $\widetilde{v}_j^{(s_*)}$ at last step s_* in view of (3.26). This is convenient as we only need to work with a fixed $\widetilde{v}_j$, namely $\widetilde{v}_j = \widetilde{v}_j^{(s_*)}$.

We first make measure estimates in $\widetilde{V}$ via (4.1), and then convert the estimates to estimates in V by using the Jacobian estimates (2.17). Denote for a given n

$$j_+(n) = \max\{j \in \mathbb{Z} | n_j - n'_j \neq 0\}.$$

The set of acceptable $\widetilde{V}$ contains

$$S = \bigcap_{||k|-j_0|<N} \ \bigcap_{s=1,\dots,s_*} \ \bigcap_{n \text{ satisfies } (4.2), j_+(n)=k} \left\{ \widetilde{V} \middle| \left| \sum_{j \leq k} (n_j - n'_j) \widetilde{v}_j \right| > \delta_s^{\frac{1}{100s^2}} \right\}. \tag{4.3}$$

Let

$$S_k = \bigcap_{s=1,\dots,s_*} \ \bigcap_{n \text{ satisfies } (4.2), j_+(n)=k} \left\{ \widetilde{V} \middle| \left| \sum_{j \leq k} (n_j - n'_j) \widetilde{v}_j \right| > \delta_s^{\frac{1}{100s^2}} \right\}.$$

Then for fixed $(\widetilde{v}_j)_{j<k}$ (note strict inequality here) and n such that $j_+(n) = k$

$$\operatorname{mes}_{\widetilde{v}_k} \left\{ \widetilde{V} \middle| \left| \sum_{j \leq k} (n_j - n'_j) \widetilde{v}_j \right| < \delta_s^{\frac{1}{100s^2}} \right\} < 2\delta_s^{\frac{1}{100s^2}}.$$

Let S_k^c be the complement of the set S_k. Its measure can be estimated as

$$\begin{aligned}\operatorname{mes} S_k^c &\le \sum_{s=1}^{s_*} \sum_{n \text{ satisfies } (4.2),\, j_+(n)=k} \operatorname{mes}_{\widetilde{v}_k}\left\{\widetilde{V} \,\middle|\, \left|\sum_{j\le k}(n_j - n_j')\widetilde{v}_j\right| < \delta_s^{\frac{1}{100s^2}}\right\} \\ &\le 2\sum_{s=1}^{s_*}\left(10\frac{\log\frac{1}{\delta_{s+1}}}{\log\frac{1}{\epsilon}}\right)^{20A}\delta_s^{\frac{1}{100s^2}}. \end{aligned} \tag{4.4}$$

Note that $\log\delta_s^{-1} \sim s\log\frac{1}{\epsilon}$, $s_* \sim A$. Then (4.4) gives that for some positive constant C

$$\operatorname{mes} S_k^c < \epsilon^{\frac{1}{CA}}.$$

Thus, we get

$$\operatorname{mes}_{\widetilde{V}} S > (1-\epsilon^{\frac{1}{CA}})^{2N} > e^{-2N\epsilon^{\frac{1}{CA}}},$$

which together with the Jacobian estimates gives that

$$\operatorname{mes}_V S > e^{-N\epsilon^{\frac{1}{40}}} e^{-2N\epsilon^{\frac{1}{CA}}} > e^{-3N\epsilon^{\frac{1}{CA}}}.$$

For fixed j_0, S corresponds to a rare event. To circumvent it, as in [5], we allows j_0 to vary in some interval $[\bar{j}_0, 2\bar{j}_0]$. Taking into account that the restriction in (4.1) only relates to $v_j|_{||j|-j_0|<N}$, we get by using independence that with probability at least

$$1-(1-e^{-3N\epsilon^{\frac{1}{CA}}})^{\frac{j_0}{2N}} > 1-\exp\left(-\frac{j_0}{2N}e^{-3N\epsilon^{\frac{1}{CA}}}\right) \tag{4.5}$$

the condition (4.1) holds for some $j_0 \in [\bar{j}_0, 2\bar{j}_0]$, where $\frac{j_0}{2N}$ is the number of independent intervals of length $2N$ in $[\bar{j}_0, 2\bar{j}_0]$.

5 Proof of Theorem 1.1

In Sects. 3 and 4, we showed that for fixed $\bar{j}_0 \in \mathbb{Z}$ large enough, there exists a $j_0 \in [\bar{j}_0, 2\bar{j}_0]$ such that with probability

$$1-\exp\left(-\frac{j_0}{2N}e^{-3N\epsilon^{\frac{1}{CA}}}\right)$$

H is symplectically transformed into H':

$$\begin{aligned} H' = &\frac{1}{2}\sum_{j\in\mathbb{Z}}\widetilde{v}_j|q_j|^2 \\ &+ \sum_{n\in\mathbb{N}^{\mathbb{Z}}\times\mathbb{N}^{\mathbb{Z}}} c(n)\prod_{\operatorname{supp} n} q_j^{n_j}\bar{q}_j^{n_j'} \\ &+ \sum_{n\in\mathbb{N}^{\mathbb{Z}}\times\mathbb{N}^{\mathbb{Z}}} d(n)\prod_{\operatorname{supp} n}|q_j|^{2n_j} \\ &+ O(\epsilon^A), \end{aligned} \tag{5.1}$$

where $\widetilde{v}_j$ are the modulated frequencies, $c(n)$ are the coefficients of non-resonant monomials, and $d(n)$ are the coefficients of resonant monomials. The coefficients $c(n)$ satisfy

$$|c(n)| < \epsilon^A, \tag{5.2}$$

if

$$\operatorname{supp} n \cap \left\{\left[-j_0 - \frac{N}{2}, -j_0 + \frac{N}{2}\right] \cup \left[j_0 - \frac{N}{2}, j_0 + \frac{N}{2}\right]\right\} \neq \emptyset.$$

Now we are in a position to complete the proof of Theorem 1.1. The coordinates $q(t) = \{q_j(t)\}_{j\in\mathbb{Z}}$ satisfy

$$i\dot{q} = \frac{\partial H}{\partial \bar{q}}. \tag{5.3}$$

Denote the new coordinates in H' by q'. Then (5.3) becomes

$$i\dot{q}' = \frac{\partial H'}{\partial \bar{q}'}. \tag{5.4}$$

We get by using (5.1) and (5.4) that

$$\begin{aligned} \frac{d}{dt}\sum_{|j|>j_0} |q_j'(t)|^2 &= 4\operatorname{Im}\sum_{|j|>j_0} \bar{q}_j'(t)\frac{\partial H'}{\partial \bar{q}_j'} \\ &= \sum_{n\in\mathbb{N}^{\mathbb{Z}}\times\mathbb{N}^{\mathbb{Z}}} c(n) \sum_{|j|>j_0} (n_j - n_j') \prod_{\operatorname{supp} n} q_j^{n_j} \bar{q}_j^{n_j'} + O(\epsilon^A). \end{aligned} \tag{5.5}$$

Recall from (2.15, 2.16) that the monomials in (5.5) satisfy

$$\Delta(n) \le 20A.$$

So, if $\operatorname{supp} n \cap \{(-\infty, -j_0) \cup (j_0, +\infty)\} \neq \emptyset$, then

$$\operatorname{supp} n \subset (-\infty, -j_0 + 20A] \cup [j_0 - 20A, +\infty),$$

which together with (5.2) implies that if $|c(n)| \ge \epsilon^A$, then

$$\operatorname{supp} n \subset \left(-\infty, -j_0 - \frac{N}{2}\right) \cup \left(j_0 + \frac{N}{2}, +\infty\right) \subset (-\infty, -j_0) \cup (j_0, +\infty). \tag{5.6}$$

The last set in (5.6) is precisely the set that is summed over in (5.5). So from (2.9),

$$\sum_{|j|>j_0} (n_j - n_j') = 0$$

for n with $|c(n)| \ge \epsilon^A$. Thus, only terms where $|c(n)| < \epsilon^A$ contribute to (5.5) and we get

$$\frac{d}{dt}\sum_{|j|>j_0} |q_j'(t)|^2 < C\epsilon^A \quad C \text{ independent of } \epsilon, \tag{5.7}$$

where we used the fact that for the terms with $|c(n)| < \epsilon^A$

$$\operatorname{supp} n \cap \left\{\left[-j_0 - \frac{N}{2}, -j_0 + \frac{N}{2}\right] \cup \left[j_0 - \frac{N}{2}, j_0 + \frac{N}{2}\right]\right\} \neq \emptyset \quad \text{and} \quad \triangle(n), \quad |n| < 20A.$$

Integrating (5.7) in t, we obtain

$$\sum_{|j|>j_0} |q'_j(t)|^2 < \sum_{|j|>j_0} |q'_j(0)|^2 + C\epsilon^A t. \tag{5.8}$$

Note that the symplectic transformation only acts on a N neighborhood of $\pm j_0$, we obtain

$$\sum_{|j|>j_0+N} |q_j(t)|^2 = \sum_{|j|>j_0+N} |q'_j(t)|^2 < \sum_{|j|>j_0} |q'_j(t)|^2,$$

which together with (5.8) gives

$$\sum_{|j|>j_0+N} |q_j(t)|^2 < \sum_{|j|>j_0} |q'_j(0)|^2 + C\epsilon^A t.$$

On the other hand, the Hamiltonian vector field:

$$\sum \frac{\partial F}{\partial \bar{q}_j}\frac{\partial}{\partial q_j} - \frac{\partial F}{\partial q_j}\frac{\partial}{\partial \bar{q}_j}$$

preserves the ℓ^2 norm. So we have

$$\sum_{|j|>j_0} |q'_j(0)|^2 = \sum_{j\in\mathbb{Z}} |q_j(0)|^2 - \sum_{|j|\le j_0} |q'_j(0)|^2 < \sum_{|j|>j_0-N} |q_j(0)|^2.$$

Choosing $\bar{j}_0$ large enough such that

$$\sum_{|j|>j_0-N} |q_j(0)|^2 < \delta.$$

Then for $t < \frac{\delta}{C}\epsilon^{-A}$

$$\sum_{|j|>j_0+N} |q_j(t)|^2 < 2\delta$$

with probability

$$1 - \exp\left(-\frac{j_0}{2N} e^{-3N\epsilon^{\frac{1}{CA}}}\right).$$

This completes the proof of Theorem 1.1 after renaming $j_0 - N$ as j_0 and $2N$ as N.

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